Towards a quantum theory of de Sitter space

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ABSTRACT: We describe progress towards constructing a quantum theory of de Sitter space in four dimensions. In particular we indicate how both particle states and Schwarzschild de Sitter black holes can arise as excitations in a theory of a finite number of fermionic oscillators. The results about particle states depend on a conjecture about algebras of Grassmann variables, which we state, but do not prove.

Contents

1. Introduction

The observed acceleration of cosmic expansion is most simply explained by positing a small positive cosmological constant (c.c.) in the low energy effective Lagrangian for the metric of four dimensional space. The existence of a positive cosmological constant raises a host of problems for theoretical high energy physics. One would like to understand the magnitude of this parameter, and the nature of quantum observables in the asymptotically de Sitter (dS) space-time that results.

With regard to the first question, Weinberg's *galactothropic* bound on the cosmological constant [1] seems to provide a satisfactory answer to the question both of the absolute magnitude and the ratio of the c.c. to the current matter density. This assumes that one has a theory in which the dark matter density ρ_0 and the strength of primordial fluctuations Q are fixed as one varies the c.c. or that Q and ρ_0 vary with Λ in such a way as to take on their real world values at the real world value of Λ. Theories in which all three parameters are independent random variables are somewhat less attractive, but can still account for a reduction of the "expected" value of the c.c. by many orders of magnitude [2].

There are two classes of meta-physical theories which are based on plausible dynamics, and could give rise to a plethora of universes with different values of the c.c. We call them meta-physical because, at least with current understanding, there is no way to make observations on the alternative universes. The more famous of the two is called the String Landscape [3], though much of it is based on ideas that are more than 20 years old and

had no connection with string theory. Holographic cosmology [4] presents a picture of a metaverse consisting of many asymptotically dS space-times embedded in a dense black hole fluid. In the Landscape, all parameters of low energy physics vary as we jump from universe to universe, while in holographic cosmology it is plausible that only the c.c. and things which depend on it in the limit when it is small, vary. There is as yet no mathematical formulation of what the Landscape is, though there have been some interesting attempts to construct one [5]. The holographic cosmology of a defect free dense black hole fluid has been mathematically formulated [4], but not the theory of the asymptotically dS defects in the fluid.

Metaphysical theories are useful, if at all, only for understanding values of low energy parameters,[1] which cannot be explained by ordinary dynamical mechanisms. To a large extent, once the parameters are chosen the metaphysical theory is of little practical relevance. This is particularly true in the holographic cosmology approach, where the structure of the theory dictates that dynamics inside each asymptotically dS bubble is independent of the outside.

From this point of view at least, the task of theoretical physics is to construct a quantum theory of a single asymptotically dS universe. It may be that such a theory will also be useful as an approximation to a future mathematical theory of the Landscape. At any rate, this is the task we will take up in the present paper. In fact, we will take up a somewhat more modest task. In the mid 80s, the focus of string theorists was on finding a Poincaré invariant description of the real world. This was not because string theorists were ignorant of the existence of cosmology, but because they imagined that the laws of particle scattering were, to a good approximation independent of both the asymptotic past and future. This project failed because no one has ever found a consistent model of quantum gravity which is Poincaré invariant in d flat spacetime dimensions without being super-Poincaré invariant. Nor have we found asymptotically AdS theories with radius large compared to the string scale in which SUSY is broken in most of the volume of AdS.

Our own take on this failure is that it indicates a close connection between the SUSY breaking we observe in the real world, and the fact that we have a positive c.c. [6] . The analog of the old string theory program is to construct a theory in eternal dS space, which will contain an approximation to particle physics where SUSY breaking is evident, but the time dependence of cosmology is neglected. If you are with us so far, we can proceed to the construction of the theory.

2. General rules of holographic space-time

In the course of constructing a holographic cosmology, the authors of [4] also invented a general formalism for the holographic description of space-time. It is motivated by simple kinematic considerations about the nature of observers in quantum mechanics, the causal structure of Lorentzian space-times, and the covariant entropy bound [7].

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[1]Here we conform to the conventional and incorrect language which identifies the c.c. as a low energy parameter. In all examples we understand, it is in fact related to the physics of the highest energy states in the theory, which are large black holes.

In quantum mechanics, *an observer* is a large quantum system with many semiclassical observables. The only way we know how to construct mathematical models of such systems is to use the rules of (possibly cut off) quantum field theory. Indeed, a field theory in finite volume is such an observer. Averages of local fields over significant fractions of the volume have very small quantum fluctuations. The tunneling amplitudes between states with different values of these macroscopic pointer variables go like e^{-cVM^d} where M is the cutoff momentum scale and d the dimension of the world volume of the field theory. The precise observations of the mathematical formalism of quantum mechanics, in which we imagine that observables of a small system can be measured with arbitrary precision, are well approximated by machines which follow the rules of quantum field theory, for VM^d a few orders of magnitude. In realistic laboratory situations this number is of order $VM^d \sim 10^{23}$.

It is an experimental fact, and follows from the rules of quantum field theory, that any such observer has a large mass, and will follow a time-like trajectory in D dimensional space-time. The rules of holographic space-time are constructed in order to describe the observations made by these time-like observers in a way that is compatible with the holographic principle. Associated with any segment of a future directed time-like trajectory, going from a point P to a point Q in its future, there is a causal diamond, consisting of the intersection of the interior of the backward light-cone of Q with that of the forward like cone of P. The boundary of this diamond is a null surface, and the maximal area space-like $D - 2$ submanifold on this surface is called the holographic screen of the causal diamond. The covariant entropy bound bounds the entropy flowing out through the future boundary of the diamond, by one quarter of the area of this screen, in Planck units. We can reconstruct the time-like trajectory from a nested sequence of causal diamonds.

In quantum mechanics entropy always refers to a particular density matrix, and Fischler and Banks argued that the only natural density matrix to choose in implementing this principle for a generic space-time is the maximally uncertain density matrix, proportional to 1. Alternatives, like a thermal density matrix, require a preferred definition of the Hamiltonian, which is anathema in a generally covariant theory. With this choice, the entropy referred to in the covariant bound is the logarithm of the dimension of the Hilbert space associated with the diamond.

There is a natural way to construct a finite dimensional Hilbert space from classical constructs associated with the diamond. Consider a small area, or pixel, on the holographic screen. The screen lies on a null surface and, on the center of this pixel there is an orthogonal null ray which penetrates the pixel. Actually there is an ambiguity here corresponding to whether the null ray is ingoing or outgoing. We should use both, but imagine that dynamics relates the incoming to outgoing rays, so we will describe only variables associated with the outgoing ones. The null direction, and the orientation of the pixel, which is a bounded region of a space-like $D - 2$ plane orthogonal to the null direction, are both encoded in a pure spinor, a solution of the Cartan-Penrose equation:

$$\bar{\psi}\gamma^\mu\psi\gamma_\mu\psi = 0. \tag{2.1}$$

The independent real solutions to this equation are quantized according to

$$[S_a, S_b]_+ = \delta_{ab}. \tag{2.2}$$

S_a, or possibly complex linear combinations of them (depending on D) transform as spinors in the tangent space of the holographic screen. More generally, we can and should enlarge the algebra of pixel operators to include information about compactified spatial dimensions [8].

The quantization of pixel operators defines an area for the pixel, via the Bekenstein-Hawking relation. Extending this to all pixels of all holographic screens of all causal diamonds of all observers, one would (over) determine the Lorentzian geometry of space-time. A finite area holographic screen would have an operator algebra

$$[S_a(m), S_b(n)]_+ = \delta_{ab}\delta_{mn}, \tag{2.3}$$

where we have exploited a Z_2 gauge invariance of the CP equation and the commutation relations to Klein transform the commutation relations between independent pixels into anti-commutation relations. This Z_2 is associated with space-time $(-1)^F$ and imposes the spin-statistics connection on our formalism.

The operators $S_a(n)$ should be thought of as a section of the spinor bundle of the holographic screen. The fact that they are finite in number (since the screen has finite area) means that we have pixelated the screen by replacing its function algebra by a finite dimensional algebra.

It is extremely interesting that the representation space of the $S_a(n)$ for fixed n is precisely that of the spin degrees of freedom of a massless superparticle with fixed momenta. So the degrees of freedom of a pixel on the holoscreen are precisely those of a massless superparticle which penetrates that pixel.

Let us now specialize to the case of four dimensional de Sitter space. It is clear that our set-up is observer dependent. In a symmetric space we are free to choose special coordinates which exhibit the symmetry. In order to describe the holographic screen of the cosmological horizon of an individual observer we want to implement the obvious SU(2) symmetry of the screen. We also have a special operator corresponding to the generator of motion H along the time-like Killing vector seen by this observer. H will be the focus of our attention in the next section.

An SU(2) invariant way of pixelating the geometry of the two-sphere is to introduce the fuzzy sphere algebra (the algebra of $N \times N$ matrices) as the definition of the topology of the sphere.[2] The spinor bundle over the fuzzy sphere is the module of rectangular $N \times N + 1$ matrices, which contains all half integral spin representations of SU(2) up to $N - \frac{1}{2}$.

The operator algebra of the cosmological horizon is thus

$$[\psi_i^A, (\psi^\dagger)_B^j]_+ = \delta_i^j \delta_B^A. \tag{2.4}$$

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[2]Gelfand's theorem shows us the functorial identity between the topology of a compact Hausdorff space and its commutative C* algebra of complex valued functions. A key idea of non-commutative geometry is to simply identify a space with every C* algebra.

The dimension of this fermionic Hilbert space is $2^{N(N+1)}$, which means that in the large N limit we should identify

$$4N^2 \ln 2 = \pi(RM_P)^2, \tag{2.5}$$

where R is the dS radius. For dS space, we believe the density matrix to be $e^{-2\pi RH}$, but we will see below that the spectrum of H is such that this estimate of the relation between R and N is unchanged for large N.

3. The static hamiltonian and the hamiltonian of Poincaré

The causal diamond of a time-like geodesic observer in dS space is covered by the static coordinate patch

$$ds^2 = -f(r)d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \tag{3.1}$$

where

$$f(r) = 1 - \frac{r^2}{R^2}.$$

In [9] one of the authors presented a qualitative ansatz for the Hamiltonian H, which generates time evolution with respect to the static time τ. The claim is that the spectrum of H is bounded by $cT_{\mathrm{dS}} = \frac{c}{2\pi R}$, where c is a constant of order one. The Hilbert space on which H acts has dimension of order $e^{\pi(RM_P)^2}$, where Newton's constant, $G = M_P^{-2}$. The spectral density of H is thus $e^{-\pi(RM_P)^2}$.

Consider, for fixed c, a random Hamiltonian with these properties, chosen from any smooth probability distribution, and a fixed initial state, $|\psi>$ in the Hilbert space. Since H is random, time averaged correlation functions in the state $|\psi>$ will approach thermal equilibrium, with an equilibrium temperature $T = k_\psi T_{\mathrm{dS}}$. We conjecture that, for large R, and generic $|\psi>$ (chosen from a uniform probability distribution on the unit sphere in Hilbert space), k_ψ will approach some average value $\bar{k}_\psi$. By adjusting c we can choose $\bar{k}_\psi = 1$. Thus we claim that the spectral characteristics assumed for H can *explain* why dS space is a thermal system with a unique temperature (see appendix for details).

This picture of the spectrum of H is supported by two other observations about dS space. The most striking is the formula for Coleman-De Luccia (CDL) tunneling probabilities between two different dS vacua. These satisfy a law of detailed balance consistent with the picture of dS space as a quantum system with a finite number of states, *but with the free energy replaced by the entropy*. This makes sense for a thermal density matrix if the spectrum of the Hamiltonian is bounded by something close to the temperature.

The second piece of evidence is that every localized object in de Sitter space decays to the vacuum. Classically the vacuum has zero energy and it makes sense to say that the real quantum vacuum is an ensemble of states with energies below T_{dS} (which vanishes in the limit $R \to \infty$). A black hole has a large classical mass, and it is inconsistent with energy conservation to say that it decays to a state with small energy. We conclude that the black hole mass cannot be close to an eigenvalue of H.

The aim of [9] was to come up with a quantum model that accounts for all of the things we think we know about dS space from the approximation called quantum field theory in curved space-time. That approximation indeed finds a thermal density matrix at the dS temperature, but with a Hamiltonian, which we will call P_0, whose eigenvalues include particle masses. QFTCST theorists think of P_0 as the generator of static time translations, but they also claim that P_0 approaches the ordinary Hamiltonian of a Poincaré invariant theory as $R \to \infty$. This is inconsistent with our model for the spectrum of H.

P_0 and H are thus different operators. P_0 is in fact only an *emergent quantity*. It describes localized excitations in a given horizon volume, which are unstable to decay (via the true H dynamics of the system) into the dS vacuum. It is useful because the time scales involved in P_0 dynamics are short compared to the decay times of the excitations. We view the eigenvalues of P_0 to be the proper description of particle masses and the masses of black holes.

To understand how the thermal ensemble with Hamiltonian H can look like the thermal ensemble with Hamiltonian P_0, we postulate a peculiar relation between the P_0 eigenvalue and the entropy deficit of the corresponding eigenspace. The calculations done in QFTCST are an approximation in which decays of most localized systems do not occur. In this approximation we do not really resolve the spectrum of H and it makes sense to take $H \approx 0$. In that case the probability in the thermal density matrix of H for having P_0 eigenvalue E is just

$$\mathrm{Tr}\ e_E e^{-\pi (RM_P)^2}, \tag{3.2}$$

where e_E is the projector on the eigenspace with $P_0 = E$. This will reproduce the probability computed from the thermal density matrix for P_0 if

$$\mathrm{Tr}\ e_E = \frac{e^{\pi(RM_P)^2 - E/T_{\mathrm{dS}}}}{\mathrm{Tr}\ e^{-P_0/T_{\mathrm{dS}}}}. \tag{3.3}$$

This relation between the Poincaré eigenvalue and the entropy deficit is valid to leading order in $\frac{M}{R(M_P)^2}$ for Schwarzschild de Sitter black holes of mass M. We view this as another piece of semi-classical evidence for the picture advanced here.

The commutator between H and P_0 has the form

$$[H, P_0] = \sum (E_j - E_i) H_{ij}, \tag{3.4}$$

where H_{ij} is the rectangular block of H with rows in the ith and columns in the jth eigenspace of P_0. The individual matrix elements in any of the H_{ij} are bounded by $\frac{c}{2\pi R}$, but most of them are much smaller than this. In particular the QFTCST claim that the effects of H dynamics look like thermal fluctuations in the thermal P_0 ensemble, tell us that matrix elements connecting huge $E_i - E_j$ difference are exponentially suppressed. The spectrum of P_0 is bounded by the Nariai black hole mass, which is of order RM_P^2.[3]

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[3]Note that black holes are not eigenstates of P_0. However, this is a bound on the energy of the decay products. For black hole masses much smaller than the Nariai mass these decay products can be captured by the static observer and might have a lifetime much longer than that of the black hole. The decay of this bound but not gravitationally collapsed system back to the dS vacuum is probably not encoded in the operator P_0, but only in H. The captured decay products could be in eigenstates of P_0

In [9] one of the authors postulated the commutation relation between these two operators to be a finite dimensional approximation to $[H, P_0] \sim \frac{1}{R} P_0$. This was motivated by the way the asymptotic Killing vectors of Minkowski space act on the dS horizon. The general considerations above show that the commutator is small but do not point to this specific form.

4. Black holes from fermionic matrices

This section is meant to replace a somewhat confused discussion in [9]. The metric of the Schwarzschild dS black hole has the same form as the static patch metric with $f(r) \rightarrow (1 - \frac{2M}{rM_P^2} - \frac{r^2}{R^2})$. This has two horizons, which are at the positive roots of

$$r^3 - rR^2 + \frac{2MR^2}{M_P^2} = 0.$$

We write this as

$$R^2 = R_+^2 + R_-^2 + R_+ R_-, \tag{4.1}$$

$$2MR^2 = R_+ R_- (R_+ + R_-) M_P^2. \tag{4.2}$$

The entropy deficit of the black hole state, taking into account both cosmological and black hole horizons, is

$$\Delta S = \pi R_+ R_- M_P^2. \tag{4.3}$$

We match the entropy of our fermionic Hilbert space to that of dS space by

$$\pi (R M_P)^2 = 4N^2 \ln 2. \tag{4.4}$$

All such formulae are to be understood only in the large N limit. In order to find candidate black hole states, we choose two integers related to $R_\pm$ by the same formula

$$\pi (R_\pm M_P)^2 = 4N_\pm^2 \ln 2. \tag{4.5}$$

We do this by choosing an integer $N_- \leq \frac{1}{\sqrt{3}} N$, and defining N_+ to be the closest integer to the solution of

$$N^2 = N_+^2 + N_-^2 + N_+ N_-, \tag{4.6}$$

satisfying the constraint

$$N_+ \geq N_-. \tag{4.7}$$

Now choose N_- rows and N_+ columns of the fermionic matrix ψ_i^A and define the black hole states with Schwarzschild radius N_- to be those annihilated by ψ_i^A for the chosen rows and columns. The reader is encouraged to think of the choice of a particular set of N_- rows and N_+ columns as analogous to the choice of a particular static coordinate system. Note for example that as N_-^2 gets large, and approaches its maximum value, $\frac{N^2}{3}$, one cannot independently choose to construct black hole states for arbitrary choices of rows and columns. We will have more to say about the way that different horizon volumes

are embedded in the index space of the fermionic matrices when we discuss particle states below.

We can reproduce the black hole mass formula 4.2 by writing

$$P_0 = \sqrt{\frac{\ln 2}{2\pi}} M_P(N^2 - 2\mathcal{N})\sqrt{N^2 - \mathcal{N}}. \qquad (4.8)$$

Here, $\mathcal{N}$ is the total fermion number operator. This formula is "coordinate invariant", in the sense that it makes no reference to the particular choice of rows and columns. The black hole states we have defined are all eigenstates of this operator, but not all with the same eigenvalue. However, for large N, the average value of P_0 in the ensemble of all black hole states is indeed the classical black hole mass, and the fluctuations in this ensemble go to zero like a power of N. We make the further rule that, when speaking of a particular horizon volume we only look at states with a particular choice of N_- rows and N_+ columns.

A puzzling feature of the fermionic formulation is that one could consider similar states with arbitrary choice of N_+ independent of N_-. This is perhaps related to black holes with angular momentum, but we have not yet studied the angular momentum properties of these states.

While the operator P_0 realizes the relation between entropy and energy that we described in the previous section, it is far from the exact Hamiltonian characterizing quantum dS space. We view it as the *asymptotic darkness* [10] approximation to that Hamiltonian, in which only black hole spectra are treated, and black holes are exactly stable. We will have to modify the Hamiltonian in order to describe black hole decay, and the particle states they decay into. It is to the second part of this task that we now turn.

5. Particles from fermionic matrices

Before embarking on this task, we should recall the extent to which physics in dS space can be described in terms of particles. We begin our analysis, *faute de mieux* with quantum field theory, though we will see below that there is a more elegant description available. Our question is: *How much of the entropy of dS space can be understood in terms of particles?* and it was answered in [11]. The entropy of quantum field theory is dominated by high energy states, and the high energy behavior of a quantum field theory is conformally invariant. The entropy of a cutoff conformal field theory in a volume of linear size R scales like

$$\Lambda_c^3 R^3,$$

where Λ_c is the momentum cutoff. A typical state in this ensemble has energy of order

$$\Lambda_c^4 R^3 < M_P^2 R,$$

where the inequality is the requirement that the Schwarzschild radius of the state be less than the dS horizon radius. This implies that the cutoff is very low

$$\Lambda_c < \sqrt{\frac{M_P}{R}}.$$

One should understand that this is not the limit on the momentum of individual particles in isolation, but only of particle belonging to the maximal entropy ensemble. Our actual description of particles in dS space will have the tradeoff between the momenta of individual particles, and the total allowed number of particles, built in to it. With this cutoff, the total field theory entropy is of order $(RM_P)^{3/2}$. In [11] the authors suggested that this counting allows us to understand the QFTCST picture of dS space as a system which (at asymptotically late or early times) has an infinite number of independent horizon volumes, each described by cutoff field theory. Our counting suggests that the number of independent field theoretic subsystems is finite, of order $(RM_P)^{1/2}$, but becomes infinite in the small c.c. limit where QFTCST is supposed to be a good approximation.

We now proceed to present our model in more detail. The holographic formulation of quantum gravity in de Sitter space we want to put forward is expected, in the $\Lambda \to 0$ limit, to recover $4d$ $\mathcal{N} = 1$ Super-Poincaré physics. At the kinematical level, we expect to recover something akin to Ashtekar's formalism for asymptotically flat spacetime at null infinity [14]. For $4d$ Minkowski space, null infinity is (u, Ω), where u is a null coordinate and Ω parameterizes a two sphere. The conformal group of the sphere is $SO(1,3)$, which is identified as the Lorentz transformations. However, as we will see below, the kinematical theory of dS space does not lead to a field theoretic formalism on null infinity. Rather, in close analogy with Matrix Theory, we obtain a direct description of multi-particle states from a theory of matrices.

To set the stage, we discuss the formulation of SUSY algebra and SUSY multiplets of $\mathcal{N} = 1$ at null infinity. First, we will show that the 4 independent solutions to the Conformal Killing spinor equation for S^2 provide a realization of the minimal supersymmetry algebra in 4d for massless multiplets. The conformal Killing spinor equation is

$$D_\mu q^{(\alpha)} = \gamma_i e_\mu^i \lambda^{(\alpha)}. \tag{5.1}$$

In the usual angular coordinates on the two sphere, the zweibein has non-vanishing components

$$e_\theta^1 = 1, \quad e_\phi^2 = \sin\theta. \tag{5.2}$$

In the representation where the two dimensional Euclidean Dirac matrices are $\sigma_{1,2}$, the spinor covariant derivatives are

$$D_\theta = \partial_\theta, \quad D_\phi = \partial_\phi - \frac{i}{2}\sigma_3 \cos\theta. \tag{5.3}$$

Four linearly independent solutions of the CKS equation are

$$q^1 = i\sqrt{1 - \cos\theta}e^{i\phi/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5.4}$$

$$q^2 = -i\sqrt{1 + \cos\theta}e^{-i\phi/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{5.5}$$

$$q^3 = \sqrt{1 + \cos\theta}e^{i\phi/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{5.6}$$

$$q^4 = \sqrt{1 - \cos\theta}e^{-i\phi/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{5.7}$$

These satisfy

$$(q^\dagger)^\alpha q^\beta = (\gamma^0\gamma^\mu)^{\alpha\beta}\hat{P}^\mu, \tag{5.8}$$

with the Weyl representation of the SO(1, 3) Dirac matrices. Here

$$\hat{P}^\mu = (1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{5.9}$$

Below we will argue that $K \times K + 1$ fermionic matrices ψ_i^A, converge as K goes to infinity to operator valued linear functionals on the space of measurable sections of the spinor bundle of the two sphere. For two sections f_a, g_a, the commutation relations are

$$[S[f], S[g]]_+ = pf_a g_a,$$

where p is a positive real number we will explain below. It follows that

$$[S[q^\alpha], S[q^\beta]]_+ = (\gamma^0\gamma^\mu)^{\alpha\beta}P_\mu.$$

We now recall that any CPT-invariant massless multiplet of $\mathcal{N} = 1$ $d = 4$ SUSY contains states with 4 helicities: $m, m - 1/2, -m + 1/2, -m$. At null infinity, these states are sections over S^2 line bundles with the corresponding charges. The charge of a line bundle is one half the power of the positive chirality spinor bundle which realizes it. This is the description of asymptotically flat $\mathcal{N} = 1$ $d = 4$ kinematic framework we aim to recover in the $\Lambda \to 0$ limit of deSitter space. Coming back to deSitter, at finite N, the S^2 sphere at null infinity is substituted by a fuzzy sphere, and the S^2 line bundle of charge m is substituted by a module, a vector space of $N \times (N + 2m)$ matrices [15]. Our main claim is this section is that we can take blocks of Grassmann variables, such that, in the limit their size goes to infinity, the 4 sections of a supermultiplet are recovered.[4]

Our proposal for the chiral multiplet, $m = 1/2$, uses the coherent state representation of the operator algebra of the variables ψ_i^A. The Hilbert space consists of functions of a Grassmann variable z_i^A, $i = 1, \ldots, p$, $A = 1, \ldots, p+1$ as a $p \times (p+1)$ matrix, with the usual Berezin inner product. As in Matrix Theory, we mean arbitrary functions of the matrix elements.

Now introduce

$$n^a \equiv z^T J_p^a z \tag{5.10}$$

and

$$N^a \equiv z J_{p+1}^a z^T, \tag{5.11}$$

where J_{2l+1}^a are the $2l + 1$ dimensional representation of the angular momentum matrices. The space of all holomorphic Grassmann functions of z decomposes into four subspaces,

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[4]Our discussion here bears some relation to that of [13]. We suspect that the relation between their framework and ours is similar to that between Hilbert spaces of gauge theories before and after fixing gauge constraints: in their Hilbert space, with dimension growing like e^{R^3}, not all degrees of freedom are simultaneously physical. After one chooses a holographic screen ("fixes the gauge"), the reduced Hilbert space has dimension e^{R^2}, as ours has.

which are of the form

$$zf_1(n^a) \tag{5.12}$$
$$f_2(n^a),$$
$$f_3(N^a),$$
$$z^T f_4(N^a).$$

In these formula, the functions f_i are *matrix polynomials*. For example, the general form of f_2 is $\sum c_{a_1 \ldots a_k} n^{a_1} \ldots n^{a_k}$, where the products are $p+1 \times p+1$ matrix products. Our claim is that every function of the matrix elements takes one of these four forms. That is, the matrix elements of these four kinds of matrix fill out the space of all functions of the matrix elements of z_i^A in a one to one fashion.

Note that the differences between the numbers of rows and columns in these four subspaces are 1,0,0,-1, as appropriate for the modules of the four sections of the $m = 1/2$ supermultiplet. The f_i are all power series in their respective matrix variables. The power series all truncate, because z is a Grassmann variable.

Our conjecture is that, one can take the $p \to \infty$ limit, in such a way that these four subspaces of the Hilbert space converge to the space of L^2 sections of four line bundles over the two sphere. This conjecture is motivated by the way in which the fuzzy sphere converges to the sphere, but has the following additional features.

- n^a and N^a are bilinears in Grassmann variables. This should become irrelevant as $p \to \infty$ because these combinations involve infinite sums of Grassmann bilinears and so all of the vanishing relations in the Grassmann algebra only come in very high order products. The L^2 sections involve sums of finite monomials in these variables (generalized spherical harmonics) with coefficients converging to zero rapidly with the order of the monomial.

- Conventionally one takes a limit of the fuzzy sphere which produces a spherical geometry with finite radius. Here we want to make the radius infinite and obtain objects which depend only on the conformal equivalence class of the geometry. The conformal group of the sphere is $SO(1,3)$, which is identified with the Lorentz transformations.

- As a consequence, wave functions will depend on a variable $p \in R^+$, a continuous limit of the discrete p. The continuous p will rescale under conformal transformations of the sphere [14, 8].

If this conjecture is correct, then our limiting single particle Hilbert space will be that of the massless chiral supermultiplet. We will realize different particles in terms of disjoint $K \times K + 1$ blocks of the $N \times N + 1$ matrix ψ_i^A. The ratios of matrix sizes for different particles take all real positive values in the limit, so we can parametrize the size of a given matrix by a positive real number p.[5] The real linear combinations of the operators

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[5]We are actually describing a limit in which the matrices become elements of the hyperfinite II_∞ factor, as in [8].

ψ_i^A and their adjoints converge to linear functionals (operator valued measures) $S_a(\mathbf{\Omega}, p)$ on the space of measurable sections of the real spinor bundle over the two sphere. The commutation relations are

$$[S[f], S[g]]_+ = p f_a g_a,$$

where $f_a(\mathbf{\Omega})$ and $g_a(\mathbf{\Omega})$ are any two sections. If we choose q_a^α to be the four real solutions of the CKS equation (linear combinations of the solutions described above), then $Q^\alpha = S[q^\alpha]$ satisfy the SUSY anti-commutation relations for a single massless superparticle with momentum $p(1, \mathbf{\Omega})$.

Our explicit construction leads directly to the chiral multiplet. Bundles on the two sphere of charge k ($2k$ is the power of the positive chirality spinor bundle) are obtained from fuzzy modules of $N \times N + 2k$ matrices. We do not see how to obtain these while simultaneously enforcing the requirement that the operators ψ converge to something that transforms in the spinor bundle.

When discussing representations of the massless SUSY algebra in four dimensions, one is similarly led most naturally to the chiral multiplet. One simply appends a phase to the transformation law of the states in order to describe higher spin massless multiplets. It is possible that one can construct a similar argument for the present system, but we do not see how to obtain this from a natural finite N construction.

One way to do it is to insist that the pixel degrees of freedom actually correspond to full 32 component spinors, and satisfy a version of the massless SUSY anticommutation relations of 11D SUGRA in the presence of central charges. Then one representation will always contain the graviton. Since we start from a holographic description of the theory, it is reasonable to assume that it will only be sensible as a dynamical theory, if it contains a graviton. If there is no quick and dirty way, like that alluded to in the preceding paragraph, to model higher helicities then we would find that our formalism only makes sense if we model four dimensional space-time as a compactification of String/M Theory. Of course, a lot more work needs to be done, before we could make such a grandiose claim.

Having dealt with single particle states, we turn to multiparticle states. The basic idea is to consider block decompositions of the full $N \times (N + 1)$ matrix, where by the previous argument, each individual block corresponds to a single particle. We take the block sizes $1 \ll p_i \ll N$ and take N and all p_i to infinity, with p_i/p_j fixed.

In particular, let's consider the following block decomposition

$$\psi = \begin{pmatrix} 1 & 2 & \dots & K \\ K & 1 & \dots & K-1 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 2 & 3 & \dots & 1 \end{pmatrix}, \tag{5.13}$$

where $K \sim \sqrt{N}$. We associate the degrees of freedom labeled by a given integer $1 \le p \le K$ with a single independent horizon volume. Note that, if we follow the hint from our black hole discussion, and treat exchanges of indices on ψ as a gauge equivalence, then the different horizon volumes are equivalent to each other. Furthermore, the different blocks in

a given horizon volume are indistinguishable, in the sense that permutations of their order is just a relabeling. As in Matrix Theory [12], we will treat this as the gauge symmetry of particle statistics.

We are free to vary the individual block sizes in a given horizon volume, but if we want to maximize the entropy, with the proviso that there be many individual particles, we should take the blocks to be approximately square, with $K \sim \sqrt{N}$ rows. If we do this, all horizon volumes are indeed treated equally. If we try to increase an individual block size to be $\gg K$, then we simultaneously squeeze out degrees of freedom in other horizons, and constrain the allowed states of particles in our own horizon volume. Thus, the idea that localized entropy in a given horizon volume is "borrowed" from the horizon, an idea which originates in the black hole entropy formula, becomes quite explicit in this construction. When we make such a large block, a description of the system in terms of black holes becomes more appropriate. We begin to see the vague outlines of a unified description of black hole and particle states, and their interactions.[6]

The natural unit for this discrete momentum, is $\frac{1}{R}$, the minimal momentum that fits inside a horizon volume. The maximal momentum for particles in our maximal entropy configuration, with block size K, is of order $\frac{\sqrt{N}}{R} \sim \sqrt{M_P/R}$ in agreement with our field theory estimate. However, unlike field theory, our formalism allows us to take individual particle momenta much larger than this, at the expense of making the momenta of other particles smaller, or reducing the total number of particles.

Using only the degrees of freedom in a single horizon (corresponding to a single integer label in our block decomposition), we can make a maximal block size of order $N^{3/4}$, which would appear to give a maximal momentum of order $(RM_P)^{-1/4}M_P$ (in the real world this would be a few TeV). However, this is not the only way to make high momentum particles. Our formalism describes particles by the way in which they register on the holographic screen at infinity. As we will see in a moment, particles of higher momentum, defined by full blocks of size $J \gg K$, have higher angular resolution on the screen. The full set of degrees of freedom in a block of size J describes superparticles whose angular wave function can be roughly any one of the first J spherical harmonics. Thus, these operators can describe of order J particles, with the same absolute value of the momentum $\frac{J}{R}(1, \boldsymbol{\Omega})$. If we only need to describe a few particles with high momentum, we can use smaller blocks, but with the wave functions "locked together". Thus, if we only use K blocks of size $K \sim \sqrt{N}$ but insist on states with exactly the same angular wave function in each block, we are describing the amplitude for a pixel in the detector on the holoscreen to absorb momentum M_P. From the point of view of scattering theory this is interpreted as a single particle with momentum M_P . Perhaps, in analogy with Matrix Theory, it should be viewed as a bound state of the particles associated with individual blocks. Note that, in contrast to Matrix theory, there is no way to talk about particles at finite separation in the kinematics we are discussing here. So the concept of a bound state does not have an obvious meaning in this context. In a similar manner, we could try to describe particles with momentum up to $(RM_P)^{1/2}M_P$ as bound states of $N^{3/2}$ blocks of size ~ 1. These

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[6]We have to admit that it is rather too vague for our taste at the moment.

would be forced into a an almost unique angular wave function, consisting of only the first few spherical harmonics. The description of particles in our formalism is thus flexible, and the number of particles and their allowed four momentum wave functions are constrained in a complicated and mutual way.

We end this section with a remark about the emergent SU(2) group of three dimensional rotations in the limiting asymptotically flat theory, which we obtain as $(RM_P) \to \infty$. It is not the same as the SU(2) group of the static observer, under which the full fermionic matrix transforms as the $[N \otimes N + 1]$. Our entire formalism was built on the hypothesis that the Poincaré Hamiltonian P_0 was a very different operator from the static Hamiltonian H. Now we see that the same is true for the rotation group.

6. Finite N corrections

If our mathematical conjecture about the limiting space of Grassman wave functions is correct, then we have isolated the kinematic degrees of freedom of particle physics from a formulation of the quantum theory of dS space, which has a finite number of states. We could then hope to get some insight about the finite c.c. corrections to particle physics observables. In particular, it is plausible that the super-Poincaré commutation relation

$$[P_\mu, Q_\alpha] = 0, \tag{6.1}$$

(with P_μ *defined*, as above, by the SUSY anticommutation relation) is not valid at finite N.

In order to describe the spectrum of particles at a given mass scale m, we should work with particles described as fermionic matrices of size (mR). It is clear from the above discussion that unless $(mR) \gg 1$, we cannot hope to obtain a description which approximates particles moving in flat space. This remark shows that *our formalism cannot describe gravitinos obeying the classical SUGRA relation $m_{3/2} \sim \frac{1}{R}$, which arises from requiring that the c.c. not look fine tuned in the low energy effective field theory sense.* This argument alone cannot fix the dependence of $m_{3/2}$ on R, but the most symmetric treatment of particles in this system uses blocks of size $N^{1/2}$. Thus one might expect corrections to SUSY degeneracies of order $N^{-1/2}$, which suggests $m_{3/2} \sim (R/M_P)^{-1/2} \sim \Lambda^{1/4}$, as conjectured in [6].

7. Conclusions

We have presented a kinematic framework for the quantum theory of de Sitter space, and identified configurations of the fundamental variables which could represent both black hole and particle states. Much remains to be done in order to develop this into a full blown theory. We list the most salient points:

- We must prove the conjecture that our finite dimensional particle Hilbert space converges to the usual Fock space of superparticles.

- We must understand how to describe compactified internal dimensions and the spectrum of non-gravitational supermultiplets. Indeed, in the limit $N \to \infty$ we expect our model to have an S-matrix which is super-Poincaré invariant, and is likely to be closely related to well understood string and M theory constructions. We might hope, e.g. that the limiting model will have an approximate description as 11D SUGRA compactified on a manifold of G2 holonomy with values of the moduli frozen at an R symmetric point. We need a kinematic description of such a compactification in terms of fermionic matrices.

- Most importantly, we need to formulate dynamical equations which determine the scattering matrix and the object which approximates it for finite N.

The first two of these desiderata seem within reach, while the third remains somewhat mysterious. One of the authors has suggested possible avenues of attack on the dynamical problem in [8].

A. Random states of large random bounded Hamiltonians have a unique temperature

In this brief appendix, we indicate how to prove the conjectures we made about random Hamiltonians. These results were explained to us by Mark Srednicki. We want to study random Hamiltonians whose spectrum is bounded between $[0, E_b]$, where we will eventually take E_b to be of order the dS temperature. Most studies refer to Hamiltonians chosen from a Gaussian random ensemble. We will assume that similar results are valid for our case.

Given such a Hamiltonian, and in the limit of a large Hilbert space, one can show that the time averaged expectation values of a class of observables converge to the expectation values in a thermal ensemble [16]. The necessary constraint on observables has to do with their matrix elements in the basis where the Hamiltonian matrix elements are Gaussian random variables.

The temperature of the thermal ensemble is related to the center of the small energy band that is allowed in the eigenbasis expansion of the initial state. It is thus determined by the expectation value of the energy in the initial state.

This expectation value is

$$\sum |a_i|^2 E_i.$$

We now want to average this over all possible initial states. The measure is the unitary invariant measure on the unit sphere in our finite dimensional Hilbert space,

$$\delta(1 - \sum |a_i|^2).$$

For large N this can be replaced by the Gaussian measure $\frac{e^{-\frac{1}{2} \sum |a_i|^2}}{Z}$. We can compute expectation values and fluctuations using Wick's theorem. The expectation value is the average eigenvalue of the Hamiltonian, and the fluctuations in this quantity as we run over

the ensemble of states is $\frac{1}{N}$. Since the energy scale is set by E_b for any smooth measure with support on the interval, this will also be the scale of the average temperature. We choose E_b so that the temperature is precisely the dS temperature. Recall that in the dS case, N is an enormously large number. For our own universe it would be $e^{10^{120}}$.

Thus, assuming that random Hamiltonians with a fixed upper and lower bound thermalize generic states, we have proven the claims in the text.

Acknowledgments

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